UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Bioengineering

Homework Problems for Chapter 6.

Please correctly order your solutions and ensure your final answer is clearly identifiable.

1. Find the eigenvalues, the eigenvectors, the eigenspaces, and state the dimension of the corresponding eigenspaces for the following. Finally, find the eigenvalue matrix $\mathbf{\Lambda}$ and eigenvector matrix $\mathbf{V}$ and be sure MATLAB gives $\mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$. To compare with MATLAB you need to be sure the eigenvectors are normalized, such that $\mathbf{V}$ is an orthonormal matrix.

   a.
$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

   b.
$$\mathbf{B} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{pmatrix}$$

   c. For $E = 9$ and $\nu = 0.5$:
$$\mathbf{C} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$$

   d.
$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

2. Find $\mathbf{B}$ from the similarity transformation $\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$ for the following:

   a.
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

   b.
$$\mathbf{A} = \begin{pmatrix} 0 & 4 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

3. In the following three-part problems, show that $\mathbf{A} = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$ is diagonalizable. Then find a diagonal matrix $\mathbf{B}$ that is similar to $\mathbf{A}$. Finally, determine the similarity transformation that diagonalizes $\mathbf{A}$.

4. Compute $\mathbf{A}^9$ using the matrix in problem 3 without multiplying 9 matrices or using MATLAB

5. Find the eigenvalues for the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 4 & -7 & 9 \\ 0 & 0 & -4 & 6 & -2 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

6. Given the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 4 \\ 2 & 3 & 3 \end{pmatrix} \qquad \mathbf{C} = \mathbf{A} + \mathbf{B}$$

Use MATLAB to compute the eigenvalues of $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$. Show that the sum of eigenvalues of $\mathbf{C}$ is equal to the sum of eigenvalues of $\mathbf{A}$ and $\mathbf{B}$.

7. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

   a. Compute the eigenvalues.
   b. Show that $det(\mathbf{A}) = \prod_i \lambda_i$
   c. Show that $tr(\mathbf{A}) = \sum_i \lambda_i$

8. For

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

   a. Compute $\mathbf{\Lambda}, \mathbf{V}$ and show that $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$
   b. Let

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

   where the rows of $\mathbf{B}$ form vectors in $\mathbb{R}^2$. Compute $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{B}$ and draw the row vectors from each multiplication step on three separate plots. Based on how the vectors change, what do you think is happening in each step? (Hint: Look at the cover of your book)

9. Suppose we have the matrix

$$\mathbf{A} = \begin{pmatrix} 8 & 0 & -8 & 0 \\ 0 & 8 & 0 & -8 \\ -8 & 0 & 8 & 0 \\ 0 & -8 & 0 & 8 \end{pmatrix}$$

This matrix can be written as a block matrix of the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} \end{pmatrix}$$

where

$$\mathbf{B} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$$

Because $\mathbf{B}$ and $\mathbf{C}$ are the same size, we can say $det(\mathbf{A}) = det(\mathbf{B} - \mathbf{C}) * det(\mathbf{B} + \mathbf{C})$. Using this information, calculate the eigenvalues of $\mathbf{A}$.

10. As discussed in class, the impulse response of a certain mechanical sensor can be found from the solution of a second-order ordinary differential equation with constant coefficients. Let the output of the sensor be $g(t)$. The following homogeneous ODE describes its function when the input to the sensor is zero,

$$\frac{d^2g}{dt^2} + k_1 \frac{dg}{dt} + k_2 g = 0 \ .$$

We showed in class that the eigenvalues of the system,

$$s_{1,2} = -\frac{k_1}{2} \pm \frac{\sqrt{k_1^2 - 4k_2}}{2} \ ,$$

tell us about the damping and frequency of the sensor; essentially how quickly it responds to an impulse input and with what temporal response profile. But the response depends on whether the eigenvalues are real or complex. When parameter $\xi = k_1/2\sqrt{k_2} > 1$, the two eigenvalues are real, so the response to an impulse input does not oscillate. It just rises and falls.When $\xi = k_1/2\sqrt{k_2} < 1$, however, the eigenvalues are now complex and the sensor output transitions into a new state; specifically one that oscillates as it rises and falls. The impulse response that applies to both situations is

$$h(t) = \frac{e^{-\Omega_0 \xi t}}{\Omega} \sin(\Omega t) \ \mathrm{step}(t) \ ,$$

where $\Omega_0 = \sqrt{k_2}$, $\xi = k_1/2\sqrt{k_2}$, and $\Omega = \Omega_0\sqrt{1 - \xi^2}$.

Use the code below to explore the operation of this sensor. In it, I set $k_2 = 1 = \Omega_0$ to reduce the variables to just one, k_1. Describe (in words) how the physical parameters k_1 and k_2 influence the eigenvalues of the system to form engineering parameters $-\Omega_0\xi$ and Ω and how engineering parameters influence the response you see in the plots. This was a question asked of one of my students interviewing for a job in industry, so your response is critical!

```
t=0:0.01:40; %h1=9.975*(exp(-(1-0.05)*t) - exp(-(1+0.05)*t));
%q=sqrt(3); h2=(1/(4*q))*(exp(-(1-q/2)*t) - exp(-(1+q/2)*t));
%plot(t,h1,t,h2,'-.');figure
xi=2;h3=1/(sqrt(1-xi^2))*exp(-xi*t).*sin(sqrt(1-xi^2)*t);plot(t,h3,'.');
hold on
xi=1.01;h4=1/(sqrt(1-xi^2))*exp(-xi*t).*sin(sqrt(1-xi^2)*t);plot(t,h4,'r.')
xi=0.7;h5=1/(sqrt(1-xi^2))*exp(-xi*t).*sin(sqrt(1-xi^2)*t);plot(t,h5,'k.')
xi=0.4;h6=1/(sqrt(1-xi^2))*exp(-xi*t).*sin(sqrt(1-xi^2)*t);plot(t,h6,'m.')
xi=0.1;h7=1/(sqrt(1-xi^2))*exp(-xi*t).*sin(sqrt(1-xi^2)*t);plot(t,h7,'g.');
hold off
```

Can a sine function have a complex argument? What the heck is that?

11. For
$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
find $\exp(\mathbf{A})$. Work it out by hand and check it using `expm(A)`.

12. For the matrix
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 0 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 1 & 3 & 5 & 7 & 0 \end{pmatrix}$$
   a. Find the basis vectors for the column and row spaces. Select basis that have integer elements only.
   b. Show the column space basis are linearly independent

13. Repeat question 12 for the matrix
$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 9 & 0 & -4 \\ -3 & 1 & 1 & 3 & 7 \\ 2 & 0 & -5 & 2 & 2 \end{pmatrix}$$

14. Consider Pathway 1 in Figure 1:
   a. Find the stoichiometric matrix
   b. Use Matlab to compute the eigenvalues and eigenvectors
   c. For any eigenvalues that indicate a homogeneous pathway, draw the path given by the corresponding eigenvectors. (Hint: there should be two)

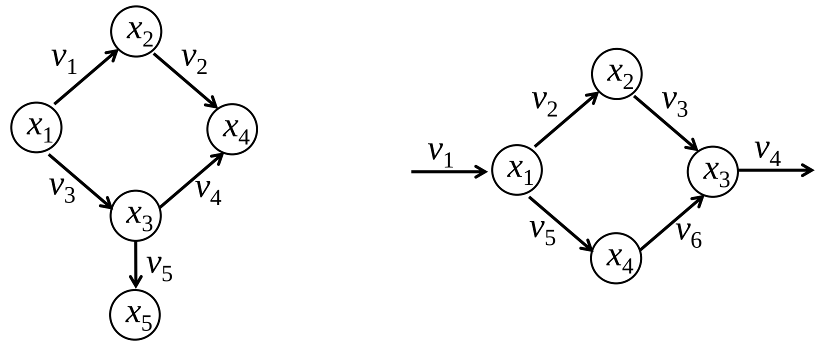

Figure 1: (left) Pathway 1 (right) Pathway 2

15. Here, Pathway 2 in Figure 1 is being evaluated.

    a. Find the possible pathways through the entire network at equilibrium ($\dot{\mathbf{x}} = 0$) by computing a basis for the solution $\mathbf{v} =?$. Draw the pathways on the graph.

    b. If you do part (a) correctly, you will see that some fluxes along the pathways are negative, which is not allowed in some situations. To fix this, left multiply the transformation matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by a matrix whose columns are the basis found in (a) and show the fluxes are now all positive. Also draw the pathways on the graph

16. Compute by hand and verify in Matlab the SVD's of the following matrices.

    a.
    $$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

    b.
    $$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$$

17. Going back to Pathway 1 in Figure 1:

    a. Use Matlab to compute the SVD of the stoichiometric matrix from Problem 14. What are the singular values?

    b. Similar to your result in Problem 14, one of the singular values should indicate a homogeneous path. What are the corresponding rows of $\mathbf{U}$ and $\mathbf{V}$ for this singular value? What do these vectors indicate?