1. Show that the set of all vectors in $\mathbb{R}^3$ of the form $a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, where $a$ is a real number, is a vector space.

2. Let $\mathbb{W}$ be a set of all $2 \times 2$ matrices having every element a positive number. Show that $\mathbb{W}$ is not a vector space.

3. (a) Show that vector $(a, 3a, 5a)$ is a subspace of $\mathbb{R}^3$. (b) Sketch this subspace on a 3-D plot.

4. Which of the following $2 \times 2$ matrices form a subspace?
   (a) The subset of all $2 \times 2$ matrices whose elements sum to six, e.g., $\begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix}$.
   (b) The subset of all $2 \times 2$ matrices having the form $\begin{pmatrix} a & a^2 \\ b & b^2 \end{pmatrix}$.

5. Determine whether vector $(4, 5, 5)$ is a linear combination of vectors $(1, 2, 3)$, $(-1, 1, 4)$, $(3, 3, 2)$.

6. (a) Is $(-1, 7)$ a linear combination of $(1, -1)$, $(2, 4)$?
   (b) Is $(-1, 15)$ a linear combination of $(-1, 4)$, $(2, -8)$?

7. Give three vectors that are linear combinations of the following pair, $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (1, 2, 0)$.

8. Determine if the first matrix is a linear combination of the others.
   (a) $\begin{pmatrix} 5 & 7 \\ 5 & -10 \end{pmatrix}$; $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$
   (b) $\begin{pmatrix} 4 & 1 \\ 7 & 10 \end{pmatrix}$; $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$
9. (a) Determine whether \( f(x) = 3x^2 + 2x + 9 \) is a linear combination of \( g(x) = x^2 + 1 \), \( h(x) = x + 3 \).

(b) Determine whether \( f(x) = x^2 + 4x + 5 \) is a linear combination of \( g(x) = x^2 + x - 1 \), \( h(x) = x^2 + 2x + 1 \).

10. (Theory) Let \( v, v_1, v_2 \) be vectors in vector space \( V \). Let \( v \) be a linear combination of \( v_1 \) and \( v_2 \). If \( c_1 \) and \( c_2 \) are nonzero scalars, show in general that \( v \) is also a linear combination of \( c_1 v_1 \) and \( c_2 v_2 \).

11. (Theory) Let \( v_1 \) and \( v_2 \) span the vector space \( V \). Let \( v_3 \) be any other vector in \( V \). Show that in general \( v_1, v_2, v_3 \) also span \( V \). (Hint: Start with \( v = a_1 v_1 + a_2 v_2 + v_3 - v_3 \), where \( v_3 = b_1 v_1 + b_2 v_2 \).)

12. If the set of vectors \( \{v_1, v_2\} \) are linearly independent, show that \( \{v_1 + v_2, v_1 - v_2\} \) are also linearly independent.

13. Which of the following sets of vectors are linearly dependent?

   (a) \( \{(2, -1, 3), (-4, 2, -6), (8, 0, 1)\} \)

   (b) \( \{(5, 2, -3), (3, 0, 4), (-3, 0, -4)\} \)

   (c) \( \{(1, 1, 1), (2, 2, 2), (0, 1, 5)\} \)

14. Find \( t \) such that the following sets are linearly dependent.

   (a) \( \{(-1, 2), (t, -4)\} \)

   (b) \( \{(2, -t), (2t + 6, 4t)\} \)

15. Are the following vectors linearly dependent? (Use relationships among elements to decide. No matrix analysis on this one.)

   \( \{(1, 2, 3), (1, 1, 1), (2, 3, 4)\} \)

16. Determine whether the following sets of matrices are linearly dependent.

   (a) \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \right\} \)

   (b) \( \left\{ \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \right\} \)

17. Show that the following set is a basis for \( \mathbb{R}^2 \) by showing it spans the space and is linearly independent.

   (a) \( \{(1, 2), (3, 1)\} \)

   (b) Is this basis orthogonal?
18. Find a basis for the following systems of equations and state whether or not they are orthonormal. Also find the rank of the system matrix.

\begin{align*}
\text{(a)} & \quad y_1 = x_1 - x_2 \\
& \quad y_2 = 2x_2 + 3x_3 \\
& \quad y_3 = -x_1 + 5x_2 - x_3 \\
\text{(b)} & \quad y_1 = x_1/\sqrt{2} + x_2/\sqrt{2} \\
& \quad y_2 = x_1/\sqrt{2} - x_2/\sqrt{2}
\end{align*}

19. Determine the rank of the following four matrices.

\begin{align*}
\text{(a)} & \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \text{(b)} & \quad \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 3 & 9 \end{pmatrix}, \quad \text{(c)} & \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(d)} & \quad \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}

20. Find bases for the subspace of $\mathbb{R}^3$ spanned by the following vectors

\begin{align*}
\text{(a)} & \quad (1, 3, 2), (0, 1, 4), (1, 4, 9) \\
\text{(b)} & \quad (1, -1, 3), (1, 0, 1), (-2, 1 - 4)
\end{align*}

21. Find bases for both row and column spaces and show the dimensions for these spaces are equal.

\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 6 \end{pmatrix}

22. Let matrix $A$ be related to the augmented matrix that is labeled $B$, where if $Ax = b$ then $B = [A|b]$. For the following, state if the systems have a single (unique) solution, many solutions, or no solutions.

\begin{align*}
\text{(a)} & \quad B \text{ is } 4 \times 5; \text{ rank}(B) = 4, \text{ rank}(A) = 4. \\
\text{(b)} & \quad B \text{ is } 3 \times 4; \text{ rank}(B) = 3, \text{ rank}(A) = 2. \\
\text{(c)} & \quad B \text{ is } 4 \times 4; \text{ rank}(B) = 2, \text{ rank}(A) = 2.
\end{align*}

23. Which of the following are orthonormal sets of vectors?

\begin{align*}
\text{(a)} & \quad \{(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}), (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})\} \\
\text{(b)} & \quad \left\{ \left( \frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}}, 0 \right), \left( -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right), \left( \frac{1}{\sqrt{32}}, -\frac{2}{\sqrt{32}}, \frac{5}{\sqrt{32}} \right) \right\}
\end{align*}