UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Bioengineering

Homework Problems for Chapter 4.

1. Compute the determinant of the following matrices using cofactors.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

2. Use cofactors and recursion to calculate $|\mathbf{A}|$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

3. Consider the three elementary matrices

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \qquad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   a. Compute the determinant of each matrix.

   b. Compute the determinant of $\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3$.

   c. Show that $|\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1| = |\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3| = |\mathbf{E}_1| \times |\mathbf{E}_2| \times |\mathbf{E}_3|$.

4. Reduce the following matrices to row echelon form (not reduced row echelon!) and compute the determinants from the reduced matrices. Use MATLAB to find the determinant of the original matrix and verify that the determinant of the row echelon form is unchanged.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 3 & 0 & 2 & 2 \end{pmatrix}$$

5. Let the columns of matrix $\mathbf{U}$ be the edges of a parallelogram, as shown in Figure 1a. In the figure, the label $\mathbf{U}_i$ is the vector created by the i^{th} column of $\mathbf{U}$.

   a. Calculate the area of the parallelogram and $|\mathbf{U}|$.

   b. Compute the determinant of the following transformation matrices

   $$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

   c. Apply the transformations to $\mathbf{U}$ to find $\mathbf{X} = \mathbf{AU}$ and $\mathbf{Y} = \mathbf{BU}$. Draw the columns $\mathbf{X}$ and $\mathbf{Y}$ on separate plots (the same way as Figure 1a).

   d. Calculate the area and determinant of $\mathbf{X}$ and $\mathbf{Y}$.

   e. Based on your results in (b) and (d), what information do you think is given by the determinant?

(a) (b)

Figure 1: Figures for Problems 5 and 6.

6. The conclusions drawn in Problem 5e may not be entirely correct. To further investigate the meaning of the determinant, let $\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Again, the columns drawn in Figure 1b form two sides of a parallelogram.

   a. Calculate the area of the parallelogram created by the columns of $\mathbf{V}$ and $|\mathbf{V}|$.

   b. Use the same transformations from Problem 5 to compute $\mathbf{X} = \mathbf{AV}$ and $\mathbf{Y} = \mathbf{BV}$. Again, draw the columns of $\mathbf{X}$ and $\mathbf{Y}$ on separate plots.

   c. Compute the areas of $\mathbf{X}$ and $\mathbf{Y}$ and their determinants.

   d. Now, compute $\frac{|\mathbf{X}|}{|\mathbf{V}|}$ and $\frac{|\mathbf{Y}|}{|\mathbf{V}|}$. Compare these ratios to $|\mathbf{A}|$ and $|\mathbf{B}|$. This time, what can you infer about the determinant?

7. Find the inverse of the following matrices using 1) an augmented matrix and Gauss-Jordan elimination and 2) the adjoint matrix and determinant.

   $$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}$$

8. One method to solve a linear system of equations $\mathbf{Ax} = \mathbf{b}$ is to use Cramer's rule. With this method, the value of x_i is found by calculating $\frac{|\mathbf{A}_i|}{|\mathbf{A}|}$, where $\mathbf{A}_i$ is the matrix $\mathbf{A}$ with the i^{th} column replaced by the vector $\mathbf{b}$. Use this method to solve the following system of equations.

$$8x_1 + x_2 + 6x_3 = 21$$
$$3x_1 + 5x_2 + 7x_3 = 22$$
$$4x_1 + 9x_2 + 2x_3 = 17$$

9. This is another convolution problem like the one given in class. As we did in class, we will study the problem as a continuous function of time and then examine it as a discrete function of time using matrix methods. Let the impulse response of a measurement system be

$$h(t') = \frac{1}{\tau} \exp\left(-\frac{t'}{\tau}\right) \times \mathrm{step}(t') \; .$$

The signal being measured by this device is a rectangular function,

$$f(t') = \mathrm{rect}\left(\frac{t'}{2T_0}\right) \; .$$

Because we have a linear time-invariant (LTI) measurement system, the output of the measurement device is given by

$$\begin{aligned} g(t) &= [h * f](t) = \int_{-\infty}^{\infty} dt' \; h(t - t') \; f(t') \\ &= \frac{1}{\tau} \int_{-\infty}^{\infty} dt' \; \exp\left(-\frac{t - t'}{\tau}\right) \times \mathrm{step}(t - t') \times \mathrm{rect}\left(\frac{t'}{2T_0}\right) \; . \end{aligned}$$

a. Compute $g(t)$. First draw the convolution components on time graphs and then figure out how to integrate the results in pieces.

b. Compute $g(t)$ using `conv(h,f)*dt` and compare your result with the analytical result of (a) by plotting both on the same graph in MATLAB .

c. Use the circulant matrix code I gave in class to compute the measurement-system matrix $\mathbf{H}$. You will need to adapt it for this problem. Then find $\mathbf{g} = \mathbf{Hf}$.

d. Can you find a way to recover $\mathbf{f}$ from $\mathbf{g}$?

Show results in four-part plot: `subplot(2,2,1);plot(t,h);hold on;plot(t,f);...` for the upper left component and `subplot(2,2,2);imagesc(H)` for the upper right corner etc. In the upper left corner, plot $\mathbf{f}$, $\mathbf{h}$, and $\mathbf{g}$ found from parts (a) and (b). In the upper right corner, generate an image of the $\mathbf{H}$ matrix from part (c). In the lower left corner, plot $\mathbf{f}$ and the $\mathbf{g}$ vector found from part c. It may not line up with the result of part (a) and that's OK. Use the lower right corner to give your response to part (d).

A copy of the code used in class for a different problem is given below. Note that it needs to be adapted to work for Problem 9 above.

```
clear all; close all;
t=0:0.01:20;c=1;u0=1/(2*c);tt=t(501:1501);
h=zeros(size(t));h(951:1051)=1;%note area=1s*1=1.
f=2*cos(2*pi*u0*t);plot(tt,f(501:1501),tt,h(501:1501),'r');hold on
g=conv(h,f,'same')*0.01;plot(tt,g(501:1501),'k');hold off


N=12;h=zeros(1,N);h(1:3)=1/6;h(10:12)=1/6;%h has area one
H(1,:)=h;
for j=2:N
    H(j,j:N)=h(1:N-j+1);H(j,1:j-1)=h(N-j+2:N);
end
imagesc(H);colormap gray; axis square
f=2*cos(2*pi*[1:12]/12)';
g=H*f;figure;plot(f);hold on;plot(g,'k');hold off
```