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Homework Problems for Chapter 3.

1. Determine the rank of each matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 1 \\ 4 & 9 & 3 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 6 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mathbf{G} = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \\ 3 & \frac{3}{2} \end{pmatrix}$$

2. Find the reduced row echelon form of each matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 4 & 3 & 3 & 2 \\ 5 & 5 & 4 & 3 \end{pmatrix}$$

3. Use an augmented matrix to find the inverse of the following matrices. Verify that the inverse matrices found are correct by computed $\mathbf{AA}^{-1}$ and $\mathbf{BB}^{-1}$.

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

4. Find the reduced row-echelon form of the following matrix. Use MATLAB to verify your answer.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 1 & 3 & -2 & 0 \\ -1 & 2 & 3 & 4 & 1 & -6 \\ 0 & 4 & 4 & 12 & -1 & -7 \end{pmatrix}$$

5. Use elementary matrices to put each matrix into reduced echelon form. Show that the product of the elementary matrices is the inverse of the matrix. Be sure to multiply the elementary matrices in the correct order!

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

6. For each equation $\mathbf{Ax} = \mathbf{b}$ below, state if there are no solutions, a unique solution, or infinitely many solutions. Explain how you determined your answers.

a.
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b.
$$\begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

c.
$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

d.
$$\begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

e.
$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 7 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

7. Use Gauss-Jordan elimination to determine if the following sets of vectors are linearly independent

a.
$$\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

b.
$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix}$$

8. Find the solution (if any) to the homogeneous equation $\mathbf{Ax} = \mathbf{0}$

a.
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \end{pmatrix}$$

b.
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}$$

9. Find the solution (if any) to the following system of equations

$$x_1 + 2x_3 + 3x_4 = 0$$
$$2x_1 + x_2 + x_3 = 0$$
$$2x_3 + 4x_4 = 0$$
$$3x_1 + x_2 + 3x_3 + x_4 = 0$$

10. Find the solution to the inhomogeneous equation $\mathbf{Ax} = \mathbf{b}$ using an augmented matrix.

a.
$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$$

b.
$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

11. Solutions for the equation $\mathbf{Ax} = \mathbf{b}$ can be solved for several $\mathbf{b}$'s simultaneously using a single augmented matrix of the form $\left(\mathbf{A} \mid \mathbf{b}_1 \quad \ldots \quad \mathbf{b}_2\right)$. Use this type of augmented matrix to simultaneously solve $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{b}_1$ and $\mathbf{b}_2$ defined below.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 2 & 2 \end{pmatrix} \qquad \mathbf{b}_1 = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \qquad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

12. Use MATLAB to find the solution to the inhomogeneous equations $\mathbf{Ax} = \mathbf{b}$ below. Solve once using the `inv` function and again using left division (i.e., `A\b`). For information on left division in MATLAB look up information for the function by typing `help mldivide`.

a.
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 \\ 0 & 4 & 6 & 0 \\ 5 & 1 & 2 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 10 \\ 27 \\ 28 \\ 42 \end{pmatrix}$$

b.
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 6 & 7 & 0 & 2 \\ 5 & 5 & 6 & 3 & 1 \\ 4 & 9 & 0 & 0 & 4 \\ 10 & 1 & 2 & 2 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 10 \\ 41 \\ 45 \\ 42 \\ 30 \end{pmatrix}$$

c.
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 14 \\ 32 \\ 50 \end{pmatrix}$$

13. a. Create the stoichiometric matrix for the reaction network shown in Figure 1a. Use MATLAB to determine the reduced echelon form of the matrix. Based on the reduced form of the matrix, what can you say about the solution to the homogeneous equation $\mathbf{Sv} = \mathbf{0}$? What are the free variables, if any?

    b. Suppose one more reaction is added to the network, v_8, as shown in Figure 1b. Create the new stoichiometric matrix and once again use MATLAB to find its reduced echelon form. Based on the reduced matrix, does a solution exist for the new homogeneous equation $\mathbf{Sv} = \mathbf{0}$? If not, explain why solutions can be found for the network in Figure 1a but not the one in 1b.

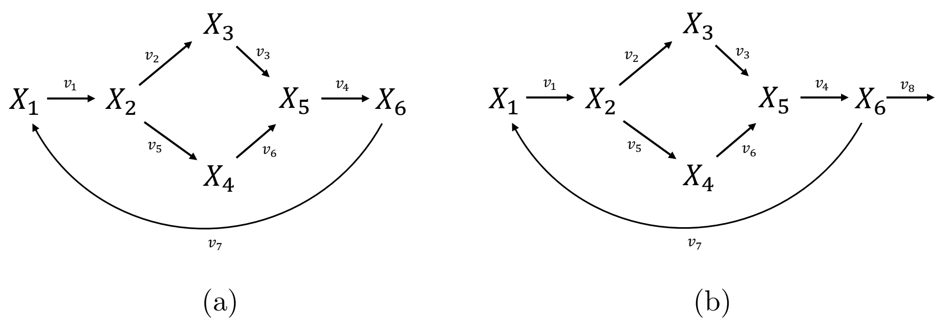

Figure 1: Reaction networks for Problem 13

14. a. Consider the following chemical equation:

$$a\ \mathrm{FeS_2} + b\ \mathrm{Cl_2} \rightarrow c\ \mathrm{FeCl_3} + d\ \mathrm{S_2Cl_2}$$

    Write the system of equations to balance the reaction. Create the matrix for the system of equations and solve the homogeneous equation $\mathbf{Ax} = \mathbf{0}$ to determine the values for a, b, c, and d.

    b. Do the same to balance the following chemical equation:

$$a\ \mathrm{Fe_2O_3} + b\ \mathrm{C} \rightarrow c\ \mathrm{Fe} + d\ \mathrm{CO_2}$$