1. The figure shows a very simple x-ray computed tomography (CT) experiment where data are acquired over four volume elements (voxels) and just two viewing angles, $\theta = 0$ and $\pi/2$. A photon flux $\Phi'$ is incident on a cube of tissue from above. The x-ray photons that make it through the cube are detected by one of two detectors that record the transmitted photon fluxes $\Phi_1$ and $\Phi_2$. For a moment, let’s focus on the $\theta = 0$ experiment on the left. Those passing through the left side of the cube are absorbed according to the attenuation coefficients $\mu_{11} + \mu_{21}$ to give us $\Phi_{1,0}$, where the subscripts 1, 0 refer to detector 1 at scanning angle 0. The equation is given by Beer’s law

$$\Phi_{1,0} = \Phi' \exp(-\mu_{11} x_1) \exp(-\mu_{21} x_2) = \Phi' \exp\left(-\sum_{i=1}^{2} \mu_{i1} x_i \right).$$

Our goal is to estimate the $\mu$’s. It is convenient to linearize the equation using

$$\varphi_{1,0} = -\ln\left[\frac{\Phi_{1,0}}{\Phi'}\right] = \sum_{i=1}^{2} \mu_{i1} x_i.$$

We complete the generalization for the $k = 1, 2$ columns to obtain the system of equations

$$\varphi_{k,0} = \sum_{i=1}^{2} \mu_{ik} x_i,$$

or in matrix form $\varphi_0 = M_0 \mathbf{x}$. 

1
This describes the entire data set of projections for acquisition at $\theta = 0$. $\varphi_0$ and $x$ are $2 \times 1$ column vectors and $M_0$ is a $2 \times 2$ matrix. Note that the arrangement of elements in $M$ depend on the scanning angle, so we need the subscript.

(a) Give the elements of matrix $M_0$.

(b) Using the derivation pattern above and the geometry in the figure, find the system of equations for $\theta = \pi/2$. Express $M_{\pi/2}$ in terms of $M_0$. (Note: $\theta = \pi/2$ in the frame of the measurement system. Also note that $x_1 = x_2$ so that $(x_1 \ x_2) = (x_2 \ x_1)$.

(c) This situation brings up a more general question about rotations of data in a plane. Let the coordinate axis for the object be co-located with the rotation axis as shown below. If a pixel of interest is located at $x = (a \ b)^t$, then where is it located in the coordinate system $x'$?

![Coordinate rotation](image)

2. When designing any diagnostic system, you can evaluate its “performance” (the ability of the system to do its job) if you know the probability density functions (pdfs) for the data vector $g$ under two conditions. Let’s set the conditions to be hypothesis $H_0$ (the patient is well) and hypothesis $H_1$ (the patient is sick). We haven’t reviewed probability theory yet, I know, but let’s see if you can compute the log-likelihood function anyway. Let $g$ be an $M \times 1$ column vector of data acquired from the measurement device under evaluation. Also let $p(g|H_i)$ be the probability of obtaining data vector $g$ for a patient that is sick. Assume the data follows a Gaussian distribution with zero mean and covariance matrix $K$ of order $M \times M$. The conditional pdfs under the two hypotheses are given by

$$p(g|H_i) = [(2\pi)^M \det K_i]^{-1/2} \exp \left( -\frac{1}{2} g^t K_i^{-1} g \right).$$
Let $\lambda(g)$ be the log-likelihood function given by

$$\lambda(g) = \ln \left( \frac{p(g|H_1)}{p(g|H_0)} \right),$$

which is a scalar value that can be compared to a threshold when we wish to evaluate the system’s performance. All I want you to do is calculate $\lambda$ and reduce it to its simplest form.

3. Matrix $Q$ is composed of complex values,

$$Q = \begin{pmatrix} G_{00} & G_{01}^* \\ G_{10} & G_{11} \end{pmatrix}. \quad \text{Also} \quad \gamma_{01} = \frac{G_{01}}{\sqrt{G_{00}G_{11}}} \quad \text{and} \quad G_{01} = G_{10}.$$

However $G_{00}$ and $G_{11}$ are real. Find $Q^{-1}$ only in terms of $G_{00}$, $G_{11}$, $\gamma_{01}$ and associated conjugates.

4. Which of the following are eigenfunction of the linear system given by

$$y[n] = \sum_{k=-\infty}^{\infty} a[k] x[n - k]$$?

(a) exp($i\omega n$) + exp($i2\omega n$), (b) $5^n$, (c) $5^n$ exp($i2\omega n$).

5. Find a set of linearly independent eigenvectors for

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$  

Do so by finding the eigenvalues, determining the number of linearly independent eigenvectors for each, and then compute the eigenvectors.

6. Show that if two matrices are similar they have the same characteristic equation. (This is to exercise your matrix algebra skills.)

7. Show that $yx^\dagger z = (x, z)y$.

8. **MATLAB Exercise.**

```
clear all;
load penny;
whos
imagesc(P);axis square; colormap(gray);
q=del2(P);imagesc(q);axis square; axis off
```

Describe what this sequence does.
9. **MATLAB Exercise.** For this next demo, go to my webpage and grab the jpeg image 100_0121.jpg, where you will find a handsome professor and his gorgeous granddaughter. If you can’t grab that file, then use another color jpeg image file.

```matlab
clear all;

x=imread('W:\temp\100_0121.jpg');

whos

imshow(x);

y=double(x);figure;

subplot(3,1,1);imagesc(x(:,:,1));axis image;axis off;
subplot(3,1,2);imagesc(x(:,:,2));axis image;axis off;
subplot(3,1,3);imagesc(x(:,:,3));axis image;axis off;

colormap(gray)
```

Describe these four images? Why is the original jpeg file a 3-D matrix?