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Homework Problems for Chapter 5.

1. Show that the set of all vectors in $\mathbb{R}^3$ of the form $a\begin{pmatrix}1\\2\\3\end{pmatrix}$, where a is a real number, is a vector space.

2. Let $\mathbb{W}$ be a set of all 2×2 matrices having every element a positive number. Show that $\mathbb{W}$ is <u>not</u> a vector space.

3. (a) Show that vector $(a, 3a, 5a)$ is a subspace of $\mathbb{R}^3$. (b) Sketch this subspace on a 3-D plot.

4. Which of the following 2×2 matrices form a subspace?

   (a) The subset of all 2×2 matrices whose elements sum to six, e.g., $\begin{pmatrix}2 & -1\\0 & 5\end{pmatrix}$.

   (b) The subset of all 2×2 matrices having the form $\begin{pmatrix}a & a^2\\b & b^2\end{pmatrix}$.

5. Determine whether vector $(4, 5, 5)$ is a linear combination of vectors $(1, 2, 3)$, $(-1, 1, 4)$, $(3, 3, 2)$.

6. (a) Is $(-1, 7)$ a linear combination of $(1, -1)$, $(2, 4)$?

   (b) Is $(-1, 15)$ a linear combination of $(-1, 4)$, $(2, -8)$?

7. Give three vectors that are linear combinations of the following pair, $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (1, 2, 0)$.

8. Determine if the first matrix is a linear combination of the others.

   (a) $\begin{pmatrix}5 & 7\\5 & -10\end{pmatrix}$; $\begin{pmatrix}1 & 2\\3 & -4\end{pmatrix}$, $\begin{pmatrix}0 & 3\\1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}$

   (b) $\begin{pmatrix}4 & 1\\7 & 10\end{pmatrix}$; $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$, $\begin{pmatrix}3 & 1\\0 & 0\end{pmatrix}$, $\begin{pmatrix}-1 & -1\\2 & 3\end{pmatrix}$

9. (a) Determine whether $f(x) = 3x^2 + 2x + 9$ is a linear combination of $g(x) = x^2 + 1$, $h(x) = x + 3$.
   (b) Determine whether $f(x) = x^2 + 4x + 5$ is a linear combination of $g(x) = x^2 + x - 1$, $h(x) = x^2 + 2x + 1$.

10. (Theory) Let $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$ be vectors in vector space $\mathbb{V}$. Let $\mathbf{v}$ be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. If c_1 and c_2 are nonzero scalars, show in general that $\mathbf{v}$ is also a linear combination of of $c_1\mathbf{v}_1$ and $c_2\mathbf{v}_2$.

11. (Theory) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ span the vector space $\mathbb{V}$. Let $\mathbf{v}_3$ be any other vector in $\mathbb{V}$. Show that in general $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ also span $\mathbb{V}$. (Hint: Start with $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_3$, where $\mathbf{v}_3 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$.)

12. If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent, show that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ are also linearly independent.

13. Which of the following sets of vectors are linearly dependent?
   (a) $\{(2, -1, 3), (-4, 2, -6), (8, 0, 1)\}$
   (b) $\{(5, 2, -3), (3, 0, 4), (-3, 0, -4)\}$
   (c) $\{(1, 1, 1), (2, 2, 2), (0, 1, 5)\}$

14. Find t such that the following sets are linearly dependent.
   (a) $\{(-1, 2), (t, -4)\}$
   (b) $\{(2, -t), (2t + 6, 4t)\}$

15. Are the following vectors linearly dependent? (Use relationships among elements to decide. No matrix analysis on this one.)

$$\{(1, 2, 3), (1, 1, 1), (2, 3, 4)\}$$

16 Determine whether the following sets of matrices are linearly dependent.
   (a) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \right\}$
   (b) $\left\{ \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} \right\}$

17. Show that the following set is a basis for $\mathbb{R}^2$ by showing it spans the space <u>and</u> is linearly independent.
   (a) $\{(1, 2), (3, 1)\}$
   (b) Is this basis orthogonal?

18. Find a basis for the following systems of equations and state whether or not they are orthonormal. Also find the rank of the system matrix.

(a) $$\begin{aligned} y_1 &= x_1 - x_2 \\ y_2 &= 2x_2 + 3x_3 \\ y_3 &= -x_1 + 5x_2 - x_3 \end{aligned}$$

(b) $$\begin{aligned} y_1 &= x_1/\sqrt{2} + x_2/\sqrt{2} \\ y_2 &= x_1/\sqrt{2} - x_2/\sqrt{2} \end{aligned}$$

19. Determine the rank of the following four matrices.

$$\text{(a)} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 3 & 9 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

20. Find bases for the subspace of $\mathbb{R}^3$ spanned by the following vectors

   (a) $(1,3,2)$, $(0,1,4)$, $(1,4,9)$
   (b) $(1,-1,3)$, $(1,0,1)$, $(-2,1-4)$

21. Find bases for both row and column spaces and show the dimensions for these spaces are equal.

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 6 \end{pmatrix}$$

22. Let matrix $\mathbf{A}$ be related to the augmented matrix that is labeled $\mathbf{B}$, where if $\mathbf{Ax} = \mathbf{b}$ then $\mathbf{B} = [\mathbf{A}|\mathbf{b}]$. For the following, state if the systems have a single (unique) solution, many solutions, or no solutions.

   (a) $\mathbf{B}$ is 4×5; rank($\mathbf{B}$) = 4, rank($\mathbf{A}$) = 4.
   (b) $\mathbf{B}$ is 3×4; rank($\mathbf{B}$) = 3, rank($\mathbf{A}$) = 2.
   (c) $\mathbf{B}$ is 4×4; rank($\mathbf{B}$) = 2, rank($\mathbf{A}$) = 2.

23. Which of the following are orthonormal sets of vectors?

   (a) $\left\{\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right),\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)\right\}$
   (b) $\left\{\left(\frac{4}{\sqrt{20}},\frac{2}{\sqrt{20}},0\right),\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{2}},\frac{1}{\sqrt{6}}\right),\left(\frac{1}{\sqrt{32}},-\frac{2}{\sqrt{32}},\frac{5}{\sqrt{32}}\right)\right\}$