UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Bioengineering

Homework Problems for Chapter 2.

1. For the following matrices

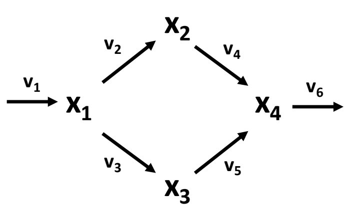
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 4 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$$

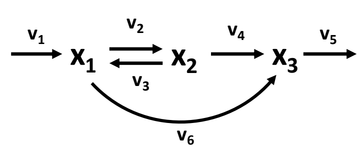
   a. Give the dimensions of each matrix
   b. Determine which matrices can be added and subtracted
   c. Determine all possible pairs of matrices (both given matrices and their respective transposes) that can be multiplied and write the dimensions of the matrix resulting from each multiplication

2. For each pathway given in Figure 1, construct the corresponding stoichiometry matrix

3. Write the mass balance equations for each pathway in Figure 1



(a) Pathway 1 for Problem 2



(b) Pathway 2 for Problem 2

Figure 1

4. Using the following matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 4 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 1 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

   a. Compute $\mathbf{A} + \mathbf{B}$
   b. Compute $\mathbf{A} - \mathbf{B}$
   c. Show that $\mathbf{AB} \neq \mathbf{BA}$
   d. Show that $\mathbf{A}^T\mathbf{B} = (\mathbf{B}^T\mathbf{A})^T$

5. Given the following matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 2 & 2 \\ 3 & 3 \\ 1 & 1 \end{pmatrix}$$

   and vectors

$$\mathbf{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   Calculate the solution to the following equations. If a solution cannot be found, explain why.

   a. $\mathbf{u}^T\mathbf{Au} + 3\mathbf{w}$
   b. $(\mathbf{AC}^T)^T\mathbf{w} - \mathbf{Cu}$
   c. $\mathbf{vBv} + \mathbf{BC}$
   d. $\mathbf{CAC}^T + 2\mathbf{B}$
   e. $\mathbf{uw}^T\mathbf{C}^T$
   f. $\mathbf{AC}^T\mathbf{Bv} + 4\mathbf{v}$

6. Use the transformation matrix $\mathbf{A}$ to compute $\mathbf{x} = \mathbf{Au}$ and $\mathbf{y} = \mathbf{Av}$. Draw $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$ on the same graph.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

7. A matrix $\mathbf{A}$ was used to transform vectors $\mathbf{u}$ and $\mathbf{v}$ into $\mathbf{x}$ and $\mathbf{y}$, respectively. Using the given values of the vectors, determine the components of matrix $\mathbf{A}$.

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 14 \\ 12 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 9 \\ 17 \end{pmatrix}$$

8. Compute $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$. Use the results to show that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{BB}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 6 & 8 \end{pmatrix}$$

9. Use the forms of $\mathbf{A}$ and $\mathbf{A}^{-1}$ given below to prove $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

10. Find the inverse of $\mathbf{A}$ by solving $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

That is, write out the full matrix product

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \longrightarrow \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and solve for a, b, c, and d.

11. A matrix raised to the k^{th} power is multiplied by itself k times, i.e.

$$\mathbf{A}^k = \underbrace{\mathbf{AAA}\ldots\mathbf{A}}_{\text{k times}}$$

   a. Prove that for an $n \times n$ matrix $\mathbf{A}$, $\mathbf{A}^k$ is also an $n \times n$ matrix.
   b. Prove $\mathbf{A}^k$ does not exist for $\mathbf{A} \in m \times n, m \neq n$.
   c. Compute $\mathbf{A}^k$ and $\mathbf{B}^k$ for $k = 2$ and $k = 3$.

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 8 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3 & 5 & 1 \\ 3 & 9 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

12. A matrix $\mathbf{A}$ is said to be idempotent if $\mathbf{A}^2 = \mathbf{A}$.

   a. Determine whether the following matrices are idempotent.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   b. Determine b, c, and d to make $\begin{pmatrix} 2 & b \\ c & d \end{pmatrix}$ idempotent.

13. A matrix is deemed nilpotent if $\mathbf{A}^p = 0$ is true for a positive integer p. The smallest value of p that makes $\mathbf{A}^p = 0$ true is called the degree of nilpotency of the matrix.

   a. Prove that the following matrices are nilpotent with degree 2.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -4 & 8 \\ -2 & 4 \end{pmatrix}$$

   b. Determine the degree of nilpotency for the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

14. Rewrite the following system of equations in the form $\mathbf{Ax} = \mathbf{b}$.

$$2x_1 + x_2 + 5x_3 + \frac{1}{2}x_4 = 3$$
$$-x_1 + 3x_2 + x_3 + 4x_4 = 0$$
$$x_1 - 2x_2 + 3x_3 - x_4 = 0$$
$$-3x_1 + \frac{3}{4}x_2 - x_3 - x_4 = 8$$

15. Rewrite the following system of equations in the form $\mathbf{Ax} = \mathbf{b}$.

$$x_1 + 4x_2 - x_5 = 3$$
$$3x_2 + 3x_3 = 1$$
$$8x_1 + 2x_6 = 0$$

16 Simplify the following matrix expressions, where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and $\mathbf{I}$ is the $n \times n$ identity matrix.

a. $\mathbf{A}(\mathbf{A} - 4\mathbf{B}) + 2\mathbf{B}(\mathbf{A} + \mathbf{B}) - \mathbf{A}^2 + 7\mathbf{B}^2 + 3\mathbf{AB}$

b. $\mathbf{B}(2\mathbf{I} - \mathbf{BA}) + \mathbf{B}(4\mathbf{I} + 5\mathbf{A})\mathbf{B} - 3\mathbf{BAB} + 7\mathbf{B}^2\mathbf{A}$

c. $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) - (\mathbf{A} + \mathbf{B})^2$

17. Determine the row operation being performed by each of the following elementary matrices. Then, determine the inverse of each matrix. Based on your findings, what rules can be used to quickly find the inverse of any given elementary matrix?

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$